ROOT, RADICAL, AND RATIONAL EXPONENT FUNCTIONS

EXAMPLE: Find the domains of the following functions.

- $f(x) = \sqrt{2x-5}$: Because of the square root, we need $2x - 5 \geq 0$ or $x \geq \frac{5}{2}$. Hence the domain is $\left[\frac{5}{2}, \infty\right)$.

- $g(x) = \dfrac{2x-1}{\sqrt[3]{x-4}}$: Because $\sqrt[3]{x-4}$ has an odd index, there are no domain restrictions due to the radical.

  However, since $\sqrt[3]{x-4}$ is in the denominator, we solve $\sqrt[3]{x-4} = 0$ to get $x = 4$ as an excluded value.

  Hence, the domain is $\{x \,|\, x \neq 4\}$ or $(-\infty, 4) \cup (4, \infty)$.

- $F(t) = (3t-1)^{\frac{3}{4}}$: Since $(3t-1)^{\frac{3}{4}} = \left(\sqrt[4]{3t-1}\right)^3$ involves an even-indexed root, we need $3t - 1 \geq 0$.

  Solving $3t - 1 \geq 0$ gives $t \geq \frac{1}{3}$, so the domain is $\left[\frac{1}{3}, \infty\right)$.

- $G(t) = \dfrac{2t}{4 - \sqrt{t+9}}$: Due to the presence of $\sqrt{t+9}$, we need $t + 9 \geq 0$, so $t \geq -9$.

  Next, we solve $4 - \sqrt{t+9} = 0$ to find any excluded values. we get $\sqrt{t+9} = 4$ so $(\sqrt{t+9})^2 = (4)^2$.

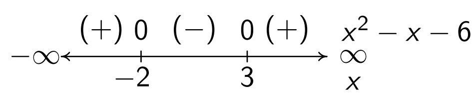
  This gives $t + 9 = 16$ or $t = 7$. Hence our domain is $\{t \,|\, t \geq -9,\ t \neq 7\}$ or $[-9, 7) \cup (7, \infty)$.

EXAMPLE: Find the domain of the following functions.

- $f(x) = \sqrt{x^2 - x - 6}$; Because of the square root, we solve $x^2 - x - 6 \geq 0$.

  To use a Sign Diagram, we first solve $x^2 - x - 6 = 0$.

  Factoring gives $(x+2)(x-3) = 0$ so $x = -2$ or $x = 3$. Our Sign Diagram is:

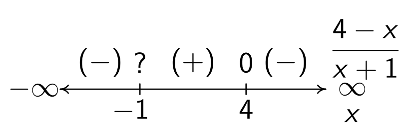


  Hence, $x^2 - x - 6 \geq 0$ on $(-\infty, -2] \cup [3, \infty)$ so this is the domain of f.

- $g(x) = \left(\dfrac{4-x}{x+1}\right)^{3/2}$: Rewriting we find $g(x) = \left(\dfrac{4-x}{x+1}\right)^{3/2} = \left(\sqrt{\dfrac{4-x}{x+1}}\right)^3$.

  Hence we need to solve: $\dfrac{4-x}{x+1} \geq 0$. We solve $x + 1 = 0$ to find $x = -1$ as the excluded value.

  We solve $\dfrac{4-x}{x+1} = 0$ to find $x = 4$ as the zero. Our Sign Diagram is:



  We find $\dfrac{4-x}{x+1} \geq 0$ on $(-1, 4]$ so this is the domain of g.

- $h(t) = 2t\,(t^2-1)^{-2/3}$: Since $(t^2-1)^{2/3} = \left(\sqrt[3]{t^2-1}\right)^2$ there are no domain restrictions from the radical.

  However, since $h(t) = 2t\,(t^2-1)^{-2/3} = \dfrac{2t}{(t^2-1)^{2/3}}$, we solve for the denominator, $(t^2-1)^{2/3} = 0$.

  We get $t^2 - 1 = 0$ or $t^2 = 1$. Hence, the excluded values are $t = \pm 1$.

  Our domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

EXAMPLE: Make a sign diagram for $f(x)=(x+2)(1-x)^{-3/2}$. We first rewrite: $f(x)=\dfrac{x+2}{(1-x)^{3/2}}$.

Our first step is to determine the domain of f. Since $(1-x)^{3/2}=\left(\sqrt{1-x}\right)^3$, we need $1-x\geq 0$ so $x\leq 1$.

Moreover, since $(1-x)^{3/2}$ is the denominator of $f(x)$, we have that $(1-x)^{3/2}\neq 0$ so $x\neq 1$.

Hence the domain of f is $\{x\,|\,x<1\}$ or $(-\infty,1)$.

To find the zeros of f, we solve $f(x)=\dfrac{x+2}{(1-x)^{3/2}}=0$. We get $x+2=0$, so $x=-2$. Our Sign Diagram is:

$$\begin{array}{cccccc} & (-) & 0 & (+) & ? & f(x) \\ -\infty \leftarrow & & -2 & & 1 & x \end{array}$$

EXAMPLE: Solve $x\,(x-3)^{-3/2}\geq 4(x-3)^{-1/2}$

As usual, our first step is to rewrite the inequality in the form $f(x)\geq 0$ and construct a Sign Diagram.

$$\begin{aligned}
x\,(x-3)^{-3/2} &\geq 4(x-3)^{-1/2} \\
x\,(x-3)^{-3/2}-4(x-3)^{-1/2} &\geq 0 \\
\frac{x}{(x-3)^{3/2}}-\frac{4}{(x-3)^{1/2}} &\geq 0 && \text{Rewrite negative exponents.} \\
\frac{x}{(x-3)^{3/2}}-\frac{4}{(x-3)^{1/2}}\cdot\frac{(x-3)^{2/2}}{(x-3)^{2/2}} &\geq 0 && \text{Get a common denominator.} \\
\frac{x}{(x-3)^{3/2}}-\frac{4(x-3)}{(x-3)^{3/2}} &\geq 0 \\
\frac{x}{(x-3)^{3/2}}-\frac{4x-12}{(x-3)^{3/2}} &\geq 0 \\
\frac{x-(4x-12)}{(x-3)^{3/2}} &\geq 0 \\
\frac{-3x+12}{(x-3)^{3/2}} &\geq 0
\end{aligned}$$

We let $f(x)=\dfrac{-3x+12}{(x-3)^{3/2}}$ and look to solve $f(x)\geq 0$ using a Sign Diagram.

Since $(x-3)^{3/2}=\left(\sqrt{x-3}\right)^3$, we need $x-3\geq 0$ or $x\geq 3$.

Moreover, in order that $(x-3)^{3/2}\neq 0$, we must have $x\neq 3$. Hence, our domain is $(3,\infty)$.

To find the zeros of f, we solve $f(x)=0$. We get $-3x+12=0$ or $x=4$.

Our Sign Diagram is:

$$\begin{array}{ccccc} ? & (+) & 0 & (-) & \\ 3 & & 4 & & \rightarrow \end{array}$$

We see $f(x)\geq 0$ on $(3,4]$, so this is our final answer.